UDC 658.51.012

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# ANALYTICAL METHODS FOR DESIGNING TECHNOLOGICAL TRAJECTORIES OF THE OBJECT OF LABOUR IN A PHASE SPACE OF STATES

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# АНАЛІТИЧНІ МЕТОДИ ПРОЕКТУВАННЯ ТЕХНОЛОГІЧНИХ ТРАЄКТОРІЙ ПРЕДМЕТІВ ПРАЦІ У ФАЗОВОМУ ПРОСТОРІ СТАНІВ

**Purpose.** The development of analytical methods for designing technological motion trajectories of objects of labour in the state space with the purpose of construction of closed PDE-models, used to describe the manufacturing system.

**Methodology.** For derivation of an equation of the labour object movement in the phase space of states, there has been applied a mathematical tool and the variational calculation methods of analytical mechanics.

**Findings.** An equation of labour object movement in the state of space has been derived and motion integrals, related to the uniformity of time and state space have been considered.

**Originality.** PDE-models of manufacturing systems, used for the engineering of the high performance manufacturing control systems have been improved. The offered model of technological resources transfer to the object of labour is based not on the traditional phenomenological description of the static production phenomena, but on conservation laws, which characterize the transfer process of technological resources to the object of labour and space-time structure of the manufacturing process. It allowed deriving the equation of the objects of labour movement along the manufacturing route, followed by the construction of non-steady-state equations of the PDE models on their ground for the description of the parameters status of the manufacturing process. While deriving the equations of the technological path of movement of the object of labour there were taken into consideration differential constraints, being applied by the manufacturing system to the transfer process of technological resources of the objects of labour, resulting from their interaction with production equipment and against each other in the course of transfer from one manufacturing operation to another.

**Practical value.** Methods for driving the equation of technological path of the object of labour allow developing high-quality models of the transfer processes of the manufacturing system, which are the basis for the high quality enterprise management system engineering with a straight flow method of industrial organization.

**Keywords:** *the object of work, manufacturing process, technological path, PDE-model*

**Introduction.** The process of the technology engineering of the product manufacturing presents a search for technological paths for the parameters of the objects of labour, defining the process of its production in accordance with construction and technological documentation [1]. A path, containing the points with values of ever changing specified parameters [1] of the objects of labour in accordance with the specified manufacturing technology is a regulatory technological path of the products manufacturing (Fig. 1) [2]. The technological paths, defining the changes of the parameters of the objects of labour, are valid (outside the rejection area), if the divergence of parameters falls within the limits of manufacturing tolerance. The selection of the regulatory technological path, which corresponds to the prescribed manufacturing method, is defined both by technico-economic factors of production, characterizing a manufacturing cost, a manufacturing cycle and a manufacturing capacity, and the social factors of production. Each operation is characterized by the equipment, required qualification of the personnel, consumption criteria and a law concerning the transfer of the resources to the object of labour. The requirements to the parameters of the object of labour are defined by the phase space field, where technological transformation of the basic material to the finished product occurs (Fig. 1) [2, 3]. The alteration of technical and economic parameters of the production requires the transfer from one production method to another with its own regulatory technological path $S_0(t)$. In the neighbourhood of the regulatory path there are located $S_j(t)$ paths of the production of $j$-object of labour ($0 \le j \le N$), being processed at the point of time $t = t^*$ with the intensity of $\mu_j(t)$ (Fig. 1). While projecting the high performance control systems, an important attention is paid to the model of the manufacturing process. Severization of the requirements to the quality of manufacturing system models initiated the development of a new type of model within the last decade, which were named PDE-models abroad. One of the difficulties of using this type of equipment is the construction of a closed equation system of the production process. As one of the problem-solution methods related to the system of equation closing there is used a constitutive equation of the manufacturing process. However, this approach, having established itself in the course of the construction of the quasistatic models of manufacturing systems, does not allow describing transitional manufacturing processes with the satisfactory accuracy. This aspect has defined the topicality of the investigation, which distinguishes the designing method of the technological paths, the equation of which may be used for the provision of the PDE-models closing [1–3].

**Analysis of the recent research and unsolved aspects of the problem of construction of the technological paths.** Let us consider the geometric locus in technological phase space, the position of which is determined by coordinate value of consecutive states of the object of labour due to technological transformational change [1–4]. Let us consider that in the analyzed $n$-dimension technological phase space there has been defined a met-

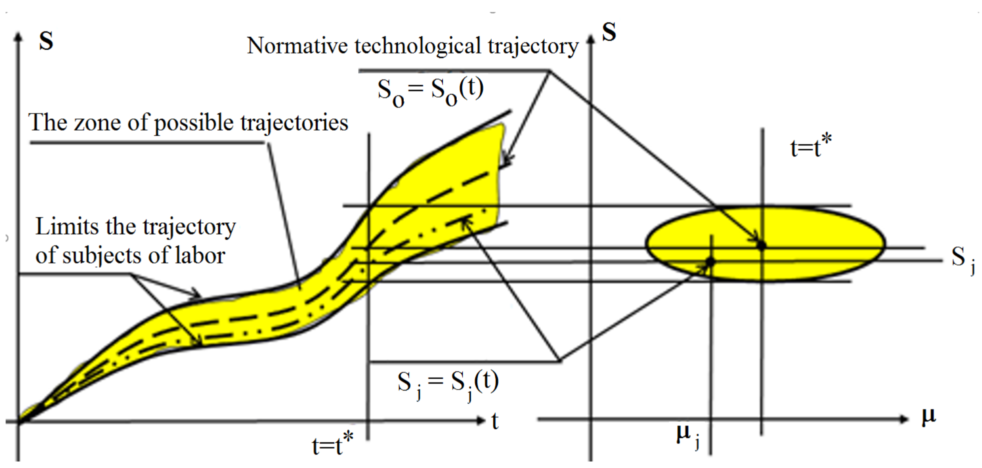


*Fig. 1. A family of technological paths. Cost S (UAH) of transferred resources to the object of labour in case of intensive processing μ (UAH/hour) depending on the total processing time t (hour)*

ric, whose square of the length element represents the following formula

$$dG^2 = \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} a_{\alpha,\beta}(q_\alpha, q_\beta)\cdot dq_\alpha \cdot dq_\beta,$$

where $a_{\alpha,\beta}(q_\alpha, q_\beta)$ are coordinate functions of technological space.

In order to analyse the manufacturing system state there is used a cost performance, which allows taking into consideration the technical and economic indicators. Herewith it is necessary to dispose a morphological structure of technological manufacturing process [5]. In order to construct a system of cost performance in general terms the morphological structure is presented as a network graph. In such a case, the cost of the finished product consists of the costs of the resources that were used: components, raw materials, materials, and labour [5], which gives an opportunity to use the square of the resources cost transferred to the subject of labour [2, 4] as a square of the length element of the phase space; this vividly characterizes the changes of the cost performance of the object of labour in the course of processing (Fig. 1) [5, 6]. There exist models which use the notions of the stage of incompleteness of the product manufacture in the capacity of variable, defining the state of the object of labour (Armbruster D., Ringhofer C., Berg V., Lefeber E.) [7–9], $x \in [0, 1]$. Such an approach is applicable to motion specification of the object of labour in one-dimension state space, which is hard to be realized when describing the product manufacturing process, in the course of which there are used several resources whose transfer to the object of labour is characterized by its own parameters. Alongside with the use of independent characteristics for the determination of the state of the object of labour, V. K. Fediukin [3] offered to use a term of the product value, which as well as a cost increases in the course of transfer from one operation to another due to processing the object of labour (Fig. 1). The resources being transferred to the object of labour are summed up together with the resources that have been transferred before while performing the previous operations. Such an operation is vividly expressed through the composition of the cost of the transferred resources [5], and is expressed by means of the following formula in the accounting enterprise balance.

The formula $dS^2 = \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} a_{\alpha,\beta}(q_\alpha, q_\beta)\, dq_\alpha dq_\beta$ is the result of the increasing natural cost of the object of labour in the course of processing by means of transferring the $\alpha$-type resources in the capacity of $dq_\alpha$ ($\alpha = 1...n$) to it. The transfer process view of the several resources of variable $x$, defining the stage of the production incompleteness of the object of labour depending on the amount of resources transferred to it $q_\alpha$, is complicated and difficult to understand. Taking into account the fact that in the course of production several kinds of technological resources are usually used, this article analyzes the technological path engineering for the object of labour in the phase space with the metric whose square of the length element is defined by means of the following formula [4]

$$dS^2 = \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} a_{\alpha,\beta}(q_\alpha, q_\beta)\cdot dq_\alpha \cdot dq_\beta. \quad (1)$$

If, while constructing the technological path, the amount of materials and raw materials $\Theta_{RaM}$ [kg], electrical energy $\Theta_E$ [kW] and the total effective time of processing $\tau_m$ [time], transferred to the object of labour are taken as the coordinates of technological space, then the state of the object of labour at the point of time t is defined by the following parameter values $q_1 = \Theta_{RaM}(t)$, $q_2 = \Theta_E(t)$, $q_3 = \tau_m(t)$, with coordinate functions, which under ordinary conditions are as follows

$$a_{\alpha,\beta}(q_\alpha, q_\beta) = Z_\alpha(q_\alpha)\cdot Z_\beta(q_\beta),$$

where $Z_1(q_1)$ [UAH/kg] is price per unit of raw material and materials; $Z_2(q_2)$ [UAH/kW] is price per electric energy unit; $Z_3(q_3)$ [UAH/hour] is price of the unit of labour time. A. V. Dabagyan offered to use the cost of components, cost value of internal activities, assembly and processing as space coordinates [5] while engineering a product.

Coordinate functions of phase technological space in most cases may be presented as the production of unit cost of technological resource or the modification unit of the technological parameter of the object of labour. The quadratic form (1) is supposed to be quite positive. The size of curve joining two points of technological path $A(q_{1A}, q_{2A}, ..., q_{\alpha A}, ..., q_{A0})$ and $B(q_{1B}, q_{2B}, ..., q_{\alpha B}, ..., q_{nB})$

$$\Delta S = \int_A^B \sqrt{\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} a_{\alpha,\beta}(q_\alpha, q_\beta)\, dq_\alpha dq_\beta},$$

represents the change in cost of the labour object when transferring from one state to another. The distance between two points of technological path of the coordinate space is the change in cost of the object of labour as a result of technological processing. In the further discourse when designing the paths, there will be used a notion of event. The event is defined by the position in the coordinate technological space, where it occurred and the time when it occurred. The typical state trajectory for the DES-models of the manufacturing systems has been considered by Ramadge P. J. and Wonham W. M. In Fig. 2 the paths of labour object motion along the technological route with discrete and continuous variations of the state parameters are presented in one dimension space. As a state parameter of the object of labour on $m$-operation in the course of the effective processing period $\Delta\tau = \tau_m - \tau_{m-1}$ there was used a value of the resource transfer intensity $\mu_{m,\psi}$ for the DES-models and $\mu_\psi(\tau)$ for the models with the continuous variation of state. Another approach when describing the continuous flow lines was presented in the works of Eekelen J. A., Lefeber E., Rooda J. E. The distinctive feature is the fact that the phase paths are constructed not for the state parameters of the object of labour but for the parameters that characterize the workstation condition (Fig. 3), which is typical for the fluid models

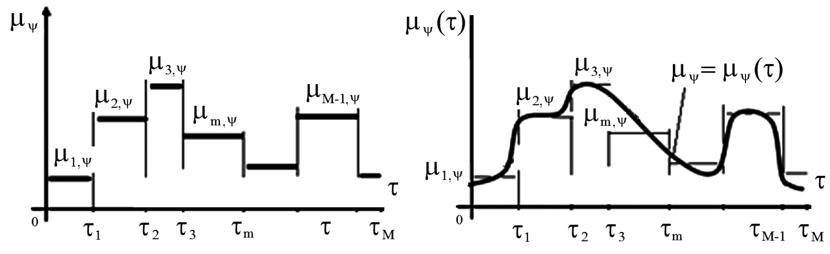


*Fig. 2. Typical path $\mu_\psi(\tau)$ for the state parameters of the object of labour:*
*a – DES model (Ramadge P. J, Wonham W. M); b – model with the continuous variation of state*

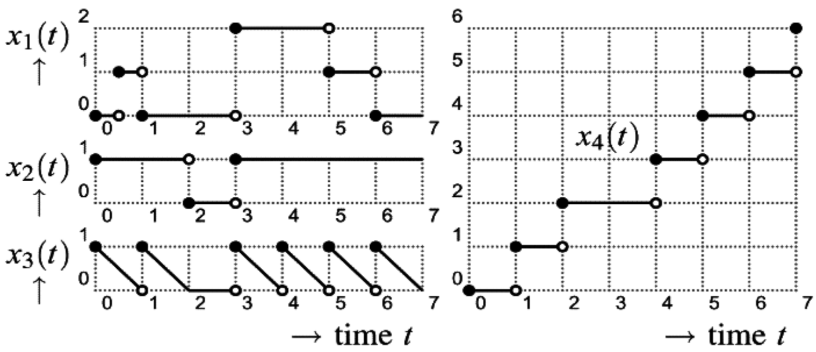


*Fig. 3. Phase paths for state parameters of the workstation (Eekelen J. A., Lefeber E., Rooda J. E):*
*$x_1(t)$ – the number of objects of work in the operational process stock; the number of objects of labour in the course of processing $x_2(t)$ with the duration of time $x_3(t)$, necessary for the accomplishment of processing; the number of objects $x_4(t)$, processed by the workstation over the period of time $[t_0, t]$*

of the manufacturing systems. As state parameters, the following variables are used: $x_1(t)$ as the number of the objects of labour in the interoperation process stock; the number of the objects of labour in the course of processing $x_2(t)$ with the duration of time $x_3(t)$, necessary for the accomplishment of processing; the number of objects $x_4(t)$, which have been processed by the work station during the particular period of time $[t_0, t]$.

**Presentation of the main research.** For the derivation of an equation of the technological path in the investigation there will be considered two events in the phase technological space (Fig. 4), which relate to the change in state of the object of labour in the course of processing. Let the first event be the fact that at the point of time $t_0$ the object of labour is under the conditions with the following coordinates ($q_{\alpha 0}$). Let the second event be the fact that at the point of time ($t_0 + dt$) the object of labour has passed to the point with the coordinates ($q_{\alpha 0} + dq_{\alpha 0}$). On the one hand, over the time of $dt$ the object of labour has travelled the path in $n$-dimension coordinate space, which equals to $\sqrt{\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} a_{\alpha,\beta}(q_\alpha, q_\beta) dq_\alpha dq_\beta}$, on the other hand over the period of $dt$ to the object of labour in the volume element, limited by the following coordinates ($q_1, q_2, \ldots, q_n$) and ($q_1 + dq_1, q_2 + dq_2, \ldots, q_n + dq_n$), there were transferred the technological resources in the amount of $\mu_\psi(t, q_1, q_2, \ldots, q_\alpha, \ldots, q_n)$. We can record the relation between the coordinates of two events under the consideration in $n$-dimension technological space

$$(\mu_\psi \cdot dt)^2 - (dS)^2 \geq 0. \tag{2}$$

The differential constraint (2) may be used for derivation of an equation of the technological path.

***1. Equation of technological relations.*** Dividing (2) into $dt$. we will receive a unilateral differential constraint

$$F_\nu\left(t, q_\alpha, \frac{dq_\alpha}{dt}\right) \geq 0$$

$$F_\nu\left(t, q_\alpha, \frac{dq_\alpha}{dt}\right) = \left(\mu_\psi\left(t, q_1, q_2, \ldots, q_\alpha, \ldots, q_n\right)\right)^2 - \\ -\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} a_{\alpha,\beta}(q_\alpha, q_\beta) \cdot dq_\alpha \cdot dq_\beta \geq 0, \quad \nu = 1 \ldots V. \tag{3}$$

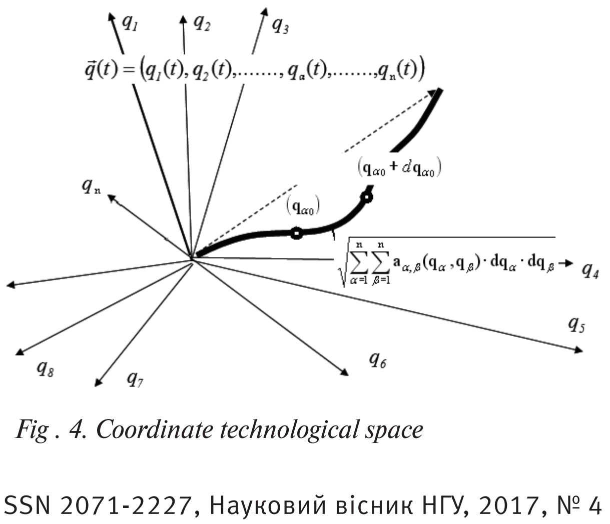

*Fig . 4. Coordinate technological space*

The labour object motion in the phase space, the unilateral constraints have been applied to, may be split in such a way that at some sections the movement of the object occurs under the unilateral constraint, while at other sections, it occurs as if this constraint does not exist at all. In such a way at separate sections, the unilateral constraint either is substituted by the bilateral constraint or is rejected. The presence of the bilateral constraint means that all the resources are being transferred to the object of labour without any loss. The technological parameters, characterizing the process of the resources transfer to the object of labour, are defined by manufacturing technology and as a rule, in the course of the products manufacture remain unchanged. It follows that the constraint (3) will not explicitly depend on time, that is $\frac{\partial F_\nu}{\partial t}=0$. Later on, we will consider the bilateral differential constraints which do not depend on time

$$F_\nu\left(q_\alpha,\frac{dq_\alpha}{dt}\right)=\left(\mu_\psi\left(q_1,q_2,\ldots,q_\alpha,\ldots,q_n\right)\right)^2-$$
$$-\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}a_{\alpha,\beta}(q_\alpha,q_\beta)\cdot dq_\alpha\cdot dq_\beta=0,\quad \nu=1\ldots V, \tag{4}$$

with the number of degrees of freedom of the manufacturing system, describing the movement of one object of labour, $(n-V)$. It is convenient to move from coordinates $q_\alpha$, describing the movement of the work object in the phase space in natural value to the coordinates of cost value $S_j$ [4]. If the coordinate functions are presented as $a_{\alpha,\beta}(q_\alpha,q_\beta)=Z_\alpha(q_\alpha)\cdot Z_\alpha(q_\alpha)$, then putting $dS_j=Z_j(q_j)\cdot dq_j$, we will obtain the bilateral differential constraints (4) through the cost coordinates $S_j$

$$F_\nu\left(S_\alpha,\frac{dS_\alpha}{dt}\right)=\left(\mu_\psi\left(S_1,S_2,\ldots,S_\alpha,\ldots,S_n\right)\right)^2-$$
$$-\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}dS_\alpha\cdot dS_\beta=0,\quad \nu=1\ldots V.$$

Having differentiated the last equation in time, we obtain the formula for the restrictions of imposing technological constraints to the movement of the object of work

$$\sum_{\alpha=1}^{n}\frac{\partial\mu_\psi}{\partial S_\alpha}\cdot\mu_\alpha-\sum_{\alpha=1}^{n}\frac{d\mu_\alpha}{dt}=0;\quad \mu_\alpha=\frac{dS_\alpha}{dt}. \tag{5}$$

For the differential bilateral constraints, which definitely depend on time the equation (5) is as follows

$$\frac{\partial\mu_\psi}{\partial t}+\sum_{\alpha=1}^{n}\frac{\partial\mu_\psi}{\partial S_\alpha}\cdot\mu_\alpha-\sum_{\alpha=1}^{n}\frac{d\mu_\alpha}{dt}=0;\quad \mu_\alpha=\frac{dS_\alpha}{dt}.$$

If the intensity of the resource transfers $\mu_\psi(S_1, S_2, \ldots, S_\alpha, \ldots, S_n)$ to the object of labour, which is presented as a sum of independent intensities of different kinds of resources $\mu_\psi\left(S_1,S_2,\ldots,S_\alpha,\ldots,S_n\right)=\sum_{\alpha=1}^{n}\mu_{\psi\alpha}\left(S_\alpha\right)$, then

$$F_\alpha\left(S_\alpha,\frac{dS_\alpha}{dt}\right)=\left(\mu_{\psi\alpha}\left(S_\alpha\right)\right)^2-\mu_\alpha^2=0,$$

which allows recording the equation of constraint for the projection to axis of reference of the cost value of $\alpha$-dimension technological resource

$$\frac{\partial\mu_{\psi\alpha}\left(S_\alpha\right)}{\partial S_\alpha}\cdot\mu_{\psi\alpha}\left(S_\alpha\right)-\frac{d\mu_\alpha}{dt}=0.$$

If the $\alpha$-dimension resource is transferred to $N_m$ of the object of labour, located in the process stock of $m$-dimension technological operation; the equation of the constraint will look as follows

$$\left(\mu_{\psi\alpha}\left(S_{\alpha,1},S_{\alpha,2},\ldots,S_{\alpha,N_m}\right)\right)^2-\sum_{k_1=1}^{N_m}\sum_{k_2=1}^{N_m}\frac{dS_{\alpha,k_1}}{dt}\frac{dS_{\alpha,k_2}}{dt}=0;$$
$$\mu_{\psi\alpha}\left(S_{\alpha,1},S_{\alpha,2},\ldots,S_{\alpha,N_m}\right)=\sum_{k=1}^{N_m}\mu_{\alpha,k}.$$

For the batch consisting of $N_m$ objects of labour, located in the interoperation process stock of $m$ operation $S_{\alpha,j}\in[S_{\psi,m},S_{\psi,m+1}[$, corresponding to the beginning and ending of the operation, let us introduce the following variables $\nu_\alpha=\frac{1}{N_m}\sum_{k=1}^{N_m}\mu_{\alpha,k}$ and $\Theta_\alpha=\frac{1}{N_m}\sum_{k=1}^{N_m}S_{\alpha,k}$, we will obtain the equation of the labour object motion of batch $N_m$ products with the averaged parameters $\nu_\alpha$ and $\Theta_\alpha$

$$\frac{\partial}{\partial\Theta_\alpha}\left(\frac{\mu_{\psi\alpha}\left(\Theta_\alpha\right)}{N_m}\right)\cdot\frac{\mu_{\psi\alpha}\left(\Theta_\alpha\right)}{N_m}-\frac{d\nu_\alpha}{dt}=0.$$

We will express the correlation of $\frac{\mu_{\psi\alpha}\left(\Theta_\alpha\right)}{N_m}=\frac{\Delta S_{\psi m}}{\Delta\tau_m}\frac{1}{N_m}$ through the density of the objects of labour $\left[\chi\right]_0(t,\Theta_\alpha)=\frac{N_m}{\Delta S_{\psi m}}$, located within the limits of the section $\Delta S_{\psi m}$ and by means of equipment work speed $\left[\chi\right]_{1_\psi}(t,\Theta_\alpha)=\frac{1}{\Delta\tau_m}$

$$\frac{\partial}{\partial\Theta_\alpha}\left(\frac{\left[\chi\right]_{1_\psi}(t,\Theta_\alpha)}{\left[\chi\right]_0(t,\Theta_\alpha)}\right)\cdot\frac{\left[\chi\right]_{1_\psi}(t,\Theta_\alpha)}{\left[\chi\right]_0(t,\Theta_\alpha)}-\frac{d\nu_\alpha}{dt}=0.$$

***2. Variation method for technical path engineering.*** The bilateral differential constraints (4) corresponding to the limiting processing case of the object of labour along the manufacturing route, when the resources are transferred to the object of labour in full, without any loss. Nevertheless, in the manufacturing activity of the company there are always certain inevitable operations, performance of which results in losses of technological resources, which is expressed by the following equation

$$F_\nu\left(q_\alpha,\frac{dq_\alpha}{dt}\right)=\left(\mu_\psi\left(q_1,q_2,\ldots,q_\alpha,\ldots,q_n\right)\right)^2-$$

$$-\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}a_{\alpha,\beta}(q_\alpha,q_\beta)\cdot\frac{dq_\alpha}{dt}\cdot\frac{dq_\beta}{dt}>0,\quad \nu=1\ldots V.$$

It is required for the technological processes of the object of labour to be fulfilled in accordance with regulatory construction and technological parameters with minimum loss of the resources. We consider that for the manufacturing system under the investigation there exist an interval, which has a minimum

$$\Re_{ab}=\int_a^b\sqrt{\left(\mu_\psi\left(q_\alpha\right)dt\right)^2-\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}a_{\alpha,\beta}(q_\alpha,q_\alpha)dq_\alpha dq_\beta},$$

for the labour object motion along the manufacturing route in accordance with the regulatory path and which is taken along the paths between two events "a" and "b" of the technological processing of the object of labour. The objective functional is presented as follows

$$\Re_{ab}=\int_a^b\left(\sqrt{\left(\mu_\psi\left(q_\alpha\right)\right)^2-\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}a_{\alpha,\beta}(q_\alpha,q_\beta)\,\frac{dq_\alpha}{dt}\frac{dq_\beta}{dt}}\right)dt,$$

with the objective function of the technological process

$$J=\sqrt{\mu_\psi^2-\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}a_{\alpha,\beta}\frac{dq_\alpha}{dt}\frac{dq_\beta}{dt}},$$

describing the behavior of the object of work in the technological process of the manufacturing system. The fact that each operation of the above-mentioned manufacturing process is characterized only by the following values $\Delta S_\psi$ (UAH) and $\mu_\psi\left(\frac{\text{UAH}}{\text{hour}}\right)$, rather than by higher derivatives, confirms the fact that the state of technological process is fully defined by the knowledge of coordinates $q_j$ (UAH) and the intensity of their change in the course of the time $\frac{dq_\alpha}{dt}\left(\frac{\text{UAH}}{\text{hour}}\right)$. From the variation that equals to zero $\delta\Re_{ab}=0$ the Euler's equations for the variational problem are derived describing the changes of the labour parameters in $n$-dimensional technological space under the effect of the equipment

$$\frac{d}{dt}\frac{\partial J}{\partial\frac{dq_\alpha}{dt}}=\frac{\partial J}{\partial q_\alpha},\quad i=1\ldots n.$$

While considering the movement of the object of labour in one dimension technological space ($n$ = 1) the objective functional may be recorded as follows

$$\Re_{ab}=\int_a^b\left(\sqrt{\mu_\psi^2-\left(\frac{dS}{dt}\right)^2}\right)dt;\quad J=\sqrt{\mu_\psi^2-\mu^2};$$

$$\mu_\psi=\mu_\psi\left(S\right);\quad \frac{dS}{dt}=\mu.$$

From the variation that equals to zero $\delta\Re_{ab}=0$ there is derived the Euler's equation

$$\frac{d\mu}{dt}=-\left(\frac{{\mu_\psi}^2-2\mu^2}{\mu_\psi}\right)\frac{\partial\mu_\psi}{\partial S}.$$

In the limiting case, when the losses of technological resources tend to zero $\mu\to\mu_\psi$ the last equation takes the form, which is similar to (5)

$$\frac{d\mu}{dt}=\mu_\psi\frac{\partial\mu_\psi}{\partial S}.\tag{6}$$

While moving along the manufacturing route the object of labour should be processed strictly in accordance with the specified manufacturing technology (Fig. 1), specified regulatory technological path $S_0=S_0(t)$ and the limiting technological paths, being set based on manufacturing tolerance for the performance of operation. Departure from the technology is inadmissible. It leads to adverse effect and results in the defect.

Let us consider the motion integrals related to uniformity of time and space. If the objective function of the manufacturing system does not depend directly on time, then its total derivative by times may be recorded as follows

$$\frac{dJ}{dt}=\sum_{\alpha=1}^{n}\frac{\partial J}{\partial q_\alpha}\cdot\frac{dq_\alpha}{dt}+\sum_{\alpha=1}^{n}\frac{\partial J}{\partial\frac{dq_\alpha}{dt}}\cdot\frac{d^2q_\alpha}{dt^2};\quad \frac{\partial J}{\partial t}=0.$$

Let us replace the derivatives $\frac{\partial J}{\partial q_\alpha}$ by $\frac{d}{dt}\frac{\partial J}{\partial\frac{dq_\alpha}{dt}}$ in accordance with the Euler's equations, we will obtain

$$\sum_{\alpha=1}^{n}\frac{d}{dt}\frac{\partial J}{\partial\frac{dq_\alpha}{dt}}-J=\text{const}.$$

This value remains unchanged through time in the course of the labour object motions in $n$-dimensional technological space, and specifies the relation between the intensity $\mu$ of the resources consumption by the object of labour and the intensity $\mu_\psi$ of the transfer of the technological resources by the manufacturing equipment. The change of the intensity $\mu$ is definitely determined by the intensity variation $\mu_\psi$. In case of uniformity of the technological space with respect to coordinate the objective function of the manufacturing system $J$ does not implicitly depend on coordinate $q_\alpha$, $\frac{\partial J}{\partial q_\alpha}=0.$ In the virtue of the Euler's equations it follows that

$$\frac{\partial J}{\partial\frac{dq_\alpha}{dt}}=\text{const}.$$

The received motion integral for the system of "the continuous flow line – the object of labour" may be treated as the constancy of the technological resources consumption rate by the object of labour in the course of its movement along the manufacturing route.

***3. Technological paths engineering with the use of general dynamic equation.*** The equation of labour object motion may be obtained from the general equation of labour object dynamics in the phase technological space.

$$\sum_{\alpha=1}^{n}\left(Q_\alpha\left(q_\alpha\right)-\sum_{\beta=1}^{n}a_{\alpha,\beta}\cdot\frac{d^2q_\beta}{dt^2}\right)\cdot\delta q_\alpha=0,$$

where $Q_\alpha(q_\alpha)$ is generalized technological strength, effecting the object of labour along the coordinate $q_\alpha$ with the purpose of transfer of the technological resources, under which there is performed work over the object of labour $\delta A_\alpha = Q_\alpha\cdot\delta q_\alpha$

$$\begin{gathered}\delta A=\sum_{\alpha=1}^{n}\delta A_\alpha;\quad \sum_{\alpha=1}^{n}\delta A_\alpha-\sum_{\alpha=1}^{n}Q_\alpha\cdot\delta q_\alpha=0;\\ \delta A-\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}a_{\alpha,\beta}\cdot\frac{d^2q_\beta}{dt^2}\cdot\delta q_\alpha=0.\end{gathered}\tag{7}$$

The summary with double amount is presented as follows

$$\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}a_{\alpha,\beta}\cdot\frac{d^2q_\beta}{dt^2}\cdot\delta q_\alpha=$$
$$=\frac{d}{dt}\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}a_{\alpha,\beta}\frac{dq_\beta}{dt}\delta q_\alpha-\delta\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\frac{a_{\alpha,\beta}}{2}\frac{dq_\beta}{dt}\frac{dq_\alpha}{dt}.$$

Let us present the derived formula in (7) and integrate in accordance with t

$$\int_{t_0}^{t_1}\left(\delta\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\frac{a_{\alpha,\beta}}{2}\cdot\frac{dq_\beta}{dt}\cdot\frac{dq_\alpha}{dt}+\delta\dot{A}\right)dt-$$
$$-\left(\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}a_{\alpha,\beta}\cdot\frac{dq_\beta}{dt}\cdot\delta q_\alpha\right)_{t=t_0}^{t=t_1}=0.$$

Since the initial and final states have been fixed, then $\delta q_i = 0$ and, consequently

$$\int_{t_0}^{t_1}\left(\delta\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\frac{a_{\alpha,\beta}}{2}\frac{dq_\beta}{dt}\frac{dq_\alpha}{dt}+\delta A\right)dt=0.$$

Let us introduce the notion of potential energy of the system $\Pi(t, q_\alpha)$, $\delta A = -\delta\Pi(t, q_\alpha)$, presenting the objective function as follows

$$J\left(t,q_\alpha,\frac{dq_\alpha}{dt}\right)=\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\frac{a_{\alpha,\beta}}{2}\frac{dq_\beta}{dt}\frac{dq_\alpha}{dt}-\Pi\left(t,q_\alpha\right).$$

If the manufacturing technology has been specified and is not changed in the course of the time or the time that has to be spent on changing such manufacturing technology is much longer than the time of the production cycle, then with the sufficient degree of accuracy one may assume the objective function does not implicitly depend on time, which allows recording the first integral

$$\sum_{\alpha=1}^{n}\frac{dq_\alpha}{dt}\cdot\frac{\partial J\left(q_\alpha,\frac{dq_\alpha}{dt}\right)}{\partial\frac{dq_\alpha}{dt}}-J\left(q_\alpha,\frac{dq_\alpha}{dt}\right)=$$
$$=\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\frac{a_{\alpha,\beta}}{2}\cdot\frac{dq_\beta}{dt}\cdot\frac{dq_\alpha}{dt}-\Pi\left(t,q_\alpha\right)=h_0=\text{const},$$

with the system potential $\Pi(q_\alpha)$. In case of one-dimension description of the manufacturing process with the objective function $J\left(S,\mu\right)=\frac{\mu^2}{2}-\Pi\left(S\right)$, the motion integral will take the following form

$$\frac{\mu^2}{2}+\Pi\left(S\right)=h_0=\text{const};\quad \frac{dS}{dt}=\mu.$$

If the motion of the object of labour complies with the bilateral constraint (5) of the kind $\mu^2-\mu_\psi^2(S)=0$, then the formula for the potential energy may be obtained from the following equality $\mu_\psi^2(S)=2(h_0-\Pi(S))$. The objective function is defined within the accuracy of the summand, which is the total derivative of the coordinate function and time. It follows thence

$$J\left(S,\mu\right)=\mu^2+\mu_\psi^2\left(S\right),$$

which allows recording the Euler's equation that coincides with the equation (6)

$$\frac{d\mu}{dt}=\mu_\psi\left(S\right)\frac{\partial\mu_\psi\left(S\right)}{\partial S}.$$

**Conclusions and recommendations for further research on development and improvement of PDE-model of the manufacturing systems.** The obtained results of the investigation are the basic ones for the development of high performance manufacturing control systems based on PDE-models of the manufacturing systems. Compared to the models, where to obtain the closed equation system the equation of condition is applied, models, which use the equation of the technological path, allow describing the manufacturing processes, functioning under the transient condition. An important and separate task is the distribution of the obtained results in case the description of the basic data have been executed in terms of fuzzy mathematics [10]. It allows extending the application field of class PDE-models of the manufacturing systems significantly.

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**Мета.** Розробка аналітичних методів проектування технологічних траєкторій руху предметів праці у просторі станів для побудови замкнутих PDE-моделей опису виробничих систем.

**Методика.** Для виведення рівняння руху предмета праці у просторі станів використані математичний апарат і методи аналітичної механіки, варіаційного числення.

**Результати.** Отримано рівняння руху предмета праці у просторі станів і розглянуті інтеграли руху, пов'язані з однорідністю часу й простору станів.

**Наукова новизна.** Полягає в удосконаленні PDE-моделей виробничих систем, використовуваних для проектування високоефективних систем управління виробництвом. Запропонована модель перенесення технологічних ресурсів на предмет праці, заснована не на традиційному феноменологічному описі стаціонарних виробничих явищ, а на законах збереження, що характеризують процес перенесення технологічних ресурсів на предмет праці й просторово-часову структуру виробничого процесу. Це дозволило отримати рівняння руху предметів праці за технологічним маршрутом з наступною побудовою на їх основі нестаціонарних рівнянь PDE-моделей для опису стану параметрів виробничого процесу. При виведенні рівняння технологічної траєкторії руху предмета праці враховані диференціальні зв'язку, що накладаються виробничою системою на процес перенесення технологічних ресурсів на предмет праці в результаті взаємодії їх з технологічним обладнанням і між собою при переході від однієї технологічної операції до іншої.

**Практична значимість.** Полягає в тому, що методи побудови рівняння технологічної траєкторії предмета праці дозволяють розробити високоточні моделі перехідних процесів виробничої системи, що є основою для проектування високоефективних систем управління підприємством з потоковим методом організації виробництва.

**Ключові слова:** *предмет праці, технологічний процес, технологічна траєкторія, PDE-модель*

**Цель.** Разработка аналитических методов проектирования технологических траекторий движения предметов труда в пространстве состояний с целью построения замкнутых PDE-моделей, применяемых для описания производственных систем.

**Методика.** Для вывода уравнения движения предмета труда в фазовом пространстве состояний использован математический аппарат и методы аналитической механики, вариационного исчисления.

**Результаты.** Получено уравнение движения предмета труда в пространстве состояний и рассмотрены интегралы движения, связанные с однородностью времени и пространства состояний.

**Научная новизна.** Заключается в совершенствовании PDE-моделей производственных систем, используемых для проектирования высокоэффективных систем управления производством. Предложена модель переноса технологических ресурсов на предмет труда, основанная не на традиционном феноменологическом описании стационарных производственных явлений, а на законах сохранения, характеризующих процесс переноса технологических ресурсов на предмет труда и пространственно-временной структуре производственного процесса. Это позволило получить уравнения движения предметов труда по технологическому маршруту с последующим построением на их основе нестационарных уравнений PDE-моделей для описания состояния параметров производственного процесса. При выводе уравнения технологической траектории движения предмета труда учтены дифференциальные связи, накладываемые производственной системой на процесс переноса технологических ресурсов на предметы труда в результате взаимодействия их с технологическим оборудованием и между собой при переходе от одной технологической операции к другой.

**Практическая значимость.** Заключается в том, что методы построения уравнения технологической траектории предмета труда позволяют разработать высокоточные модели переходных процессов производственной системы, которые являются основой для проектирования высокоэффективных систем управления предприятием с поточным методом организации производства

**Ключевые слова:** *предмет труда, технологический процесс, технологическая траектория, PDE-модель*

*Рекомендовано до публікації докт. техн. наук Л. Г. Раскіним. Дата надходження рукопису 11.06.16.*